THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. X. JUNE-JULY, 1903. NOS. 6-7.

THE APOLLONIAN PROBLEM IN SPACE.

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The so-called Apollonian problem, namely, the construction of a circle tangent to three given circles in a plane, is among the most famous in the domain of elementary geometry. It was proposed (and probably solved) by Apollonius of Perga, in one of his last works, *On Contact;* but the first solution that has come down to us is that given by Vieta in his attempted restoration which he entitled *Gallus Apollonius* (1600).* Since then it has been treated in an almost endless variety of ways, for instance, by Gergonne, Pluecker, Steiner, Hart, Casey, and recently in an elaborate memoir by Study (*Mathematische Annalen*, vol. 49).

The analagous problem in solid geometry is usually taken to be the construction of a sphere tangent to four given spheres.† There is, however, a different extension, which is the subject of this note, namely, the construction of a circle tangent to two given circles in general position in space. That this is a definite problem is seen most simply by an enumeration of constants. A circle in space requires six parameters for its determination, for example, three to fix its plane, two to fix its center in the plane, and one to fix its radius. The number of circles tangent to a given circle is ∞^3; for there are ∞^2 points on the circle, at each of which there are ∞^2 tangent planes, and in each of these planes there are ∞^1 tangent circles. Hence, tangency of circles in space is a three-fold con-

*In response to Vieta's proposal of the problem, a solution was given by Romanus which is however unsatisfactory, as it makes use of conic sections instead of the straight edge and compass only. See Cantor, *Geshichte der Mathematik*, 2nd edition, Vol. II, page 590.

†This was first considered by Fermat in his *De contactibus sphaericis*. See Cantor, Vol. II, page 659.

dition, instead of a simple condition as it is for circles in a plane or spheres in space. If a circle is to be tangent to two given circles, this is equivalent to the imposition of six simple conditions, so that the number of solutions is finite. It will be seen, in fact, that there are four solutions.

Denote the two given circles by C' and C'', their planes by π' and π'', and the line in which the latter intersect by l. The first step in the solution is to construct the sphere S which cuts both C' and C'' othogonally. The center of S must obviously lie in both π' and π'', and hence in l. Moreover, it must be so situated that the tangents from it to C' are equal to those drawn to C''. Hence the center may be obtained as follows: Revolve the circle C'' about the line l as axis until it falls in the plane π'; denote the circles so obtained by F; construct the radical axis of C' and F; this will intersect l in the point P required. For the radical axis is the locus of points for which the tangents drawn to the circles C' and F are equal; and every point in l has the property that the tangents drawn from it to C'' and F are equal. The sphere S has its center at P, and its radius is equal to either of the tangents from P to C' or C''. This construction also shows that S is unique.

Denote the two points in which C' intersects S by A', B'; and similarly the points in which C'' intersects S by A'', B''. Select one point from the first pair, and one from the second pair. Through the two points selected there passes a unique circle* orthogonal to S. This circle will be tangent to both C' and C''. In this way we obtain four circles touching both of the given circles, corresponding to the four possible pairs of points, A', A''; A', B''; B', A''; B' B''.

It remains to be shown that there are no additional solutions of the problem. Let C be any circle tangent toC' and C'' at, say, the points P' and P'' respectively. Construct the tangent lines to C at P' and P'', and denote their point of intersection by Q. With Q as center and radius $QP'=QP''$ describe a sphere. This sphere will be orthogonal to C at P' and P'', hence it is also orthogonal to C' and C''. From what precedes it follows that the sphere constructed coincides with the orthogonal sphere S. Therefore P' must lie at either A' or B', and P'' must lie at either A'' or B'', so that C necessarily coincides with one of the four circles constructed above.

The two given circles and the four circles tangent to them form an interesting set of six circles. To bring out the symmetry of the configuration, it will be convenient to introduce a new notation. Let the four points of intersection on the sphere S be denoted by P_1, P_2, P_3, P_4. Each of the six circles intersects S in two of these points, and conversely to any pair of points there corresponds one of the circles. Denote the circle corresponging to P_i, P_k by C_{ik}. The circles fall naturally into three pairs:

(I)	C_{12}, C_{34};
(II)	C_{13}, C_{24};
(III)	C_{14}, C_{23}.

*To obtain this circle, construct the great circle on S determined by the two points; the required circle then lies in the plane of this great circle and is orthogonal to it at the same two points.

The two circles of any of these pairs are in general position; the four circles tangent to them are those belonging to the other two pairs. Hence any one of the six circles is touched by four of the remaining circles, namely, by all except the one belonging to the same pair.

Pass a circle through any three of the four points of intersection, say P_2, P_3, P_4. This circle determines a unique sphere, passing through it and orthogonal so S, which may be denoted by S_1. This sphere, it is easily shown, passes through the circles C_{23}, C_{24}, C_{34}. In all there are four spheres, S_1, S_2, S_3, S_4 of this kind, each containing three of the six circles.

Let us consider the different ways in which a triple of circles can be selected.

1°. Two circles of the same pair and any other circle, for example, C_{12}, C_{34}, C_{13}; two of the circles are then in general position and the third touches both.

2°. One from each of the three pairs, but in such a way that the three circles have no common point, for example, C_{12}, C_{13}, C_{23}; the circles are then co spherical (lying on one of the spheres S_1, S_2, S_3, S_4) and are mutually tangent.

3°. Three having a point in common, for example, C_{12}, C_{13}, C_{14}; the circles are then tangent to one another at the common point.

The numbers of triples for the three types are respectively 12, 4 and 4. If we consider in each case the residual triple, for 1° it is of the same type; while for 2° it is of type 3° and *vice versa.*

The problem considered belongs essentially to the geometry based upon the group of all conformal transformations in space, namely, the group generated by inversions or transformations by reciprocal radii vectors. From the point of view of this geometry the infinite region of space is regarded as a point, instead of a plane, as in projective geometry; a straight line is thought of as a special circle, namely, one which passes through the infinite point; and similarly a plane is a special sphere.

In the preceding discussion of the problem, it was assumed that the two given circles were in general position. The enumeration of all the special cases is not difficult and makes an interesting exercise, It will suffice to state that, if the two circles intersect at a single point (including, for instance, the case where the circles become two straight lines in general position), there is no proper solution (all the four circles shrinking up into the point of intersection). On the other hand, if the circles have two real or imaginary points in common, *i. e.*, if they are co-spherical, the problem becomes indeterminate, for then there are an infinite number of tangent circles just as in the case of two circles in plane geometry.

Columbia University, *May 7, 1903.*